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QUASI-CLASSICAL APPROACH IN SUPERSYMMETRIC QUANTUM MECHANICS

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ABSTRACT

First, we briefly review supersymmetric (SUSY) quantum mechanics. Then we use a stationary phase approximation to calculate a SUSY path integral, and derive the quantization formula of Comtet, Bandrauk, and Campbell. As a by-product, we obtain a WKB-like quantization rule applicable when SUSY is broken. The broken-SUSY formula is examined for a couple of examples and found as good as the standard WKB formula.

Enter the gate to the realm of no end,And play in the field of no limit.
- Chuang Tzu -

I. INTRODUCTION

Supersymmetry (SUSY) is a fascinating idea[1]. SUSY is based on the expectation that there are environments where the distinction between bosons and fermions are irrelevant. SUSY presumes a scheme in which bosons and fermions are interchangeable and are represented by a multiplet. SUSY has been playing an important role in grand unification attempts[2]. However, so far, there has been no experimental evidence for the existence of particles necessitated by SUSY-multiplets. There are some indications that the SUSY scheme might work in understanding low energy phenomena[3]. Nevertheless, it is an observed fact that SUSY is generally broken. If SUSY is a physically meaningful symmetry in a certain environment, then a question arises as to how it would break down. In an effort to understand the possible breakdown at the level of the ordinary high energy physics, Witten[4] proposed a simple mechanism that SUSY is a good symmetry at the tree diagram level (i.e. the classical level) but spontaneously broken by small non-perturbative quantum fluctuations, and suggested to study supersymmetric quantum mechanics as a simple model. In fact, the non-perturbative breaking of SUSY in supersymmetric quantum mechanics has been well studied[5,6]. Although SUSY quantum mechanics was born as a model to simulate dynamical breaking of SUSY in field theories, rather independently of its original intent, it has become a legitimate method in non-relativistic quantum mechanics.

SUSY quantum mechanics, which is closely related to the factorization method[7,8], has been successful in solving a number of quantum mechanical problems [9,10,11]. It provides exact solutions of the Schrödinger equation for a class of so-called shape invariant potentials[8]. Such a problem as the Kaluza-Klein monopole[12] has also been solved exactly by means of the SUSY quantization method. Even for the studies of quantum tunneling[13] and anyon physics, SUSY quantum mechanics has exhibited its power.

In recent years, the quasi-classical approximation of SUSY quantum mechanics has attracted considerable attention. In 1985, Comtet, Bandrauk and Campbell[14] proposed a supersymmetric quasi-classical formula similar to the well-known WKB formula. This quantization formula, which we shall now on refer to the CBC formula, provides exact quantum spectra for all shape invariant potential systems [15]. This is in contrast to the standard WKB approach for which the Langer-like

ad hoc modifications are necessary to obtain exact results for most of the shape invariant potentials. Even for non-shape invariant systems, the CBC formula seems to yield better results than WKB does [16]. Furthermore, divergent-free wave functions can be constructed within the supersymmetric quasi-classical approximation scheme[17,18].

It has been extensively discussed how good the CBC formula is, and it is indeed true that the CBC formula has been surprisingly successful. Nonetheless, there have been little discussions as to how the formula can be derived. It has been justified mainly in conjunction with the standard WKB formula. Its correctness is judged on the basis of the smallness of the difference between the CBC and the WKB formula. However, the WKB formula is often subjected to the Langer-like modification of parameters involved. It is certainly desirable to find a way to derive the CBC formula as an approximation from the first principle. As the standard WKB quantization formula can be obtained from Feynman's path integral in the semi-classical approximation, it is natural to expect that the CBC formula may be found from a supersymmetric path integral in a similar limit. The purpose of this article is indeed to show how the CBC formula can be derived from a path integral proper to SUSY quantum mechanics. Although we wish to adopt the stationary phase method, we are not necessarily able to deal with real classical orbits for a supersymmetric quantum system. Therefore, we shall use the adjective "quasi-classical" rather than "semi-classical" for our stationary phase treatment.

Section 2 reviews SUSY quantum mechanics in a way pertinent to our discussions. In Section 3, we look into some properties of the CBC formula in comparison with WKB formula. In Section 4, we use a quasi-classical approach to calculate a bosonic path integral with the fermionic correction. The CBC formula is derived when SUSY is not broken. As a by-product, we obtain another WKB-like quantization formula when SUSY is broken. The quasi-classical wave functions are also obtained for the cases of good SUSY and broken SUSY. In Section 5, we examine the new broken SUSY formula by applying it to an exactly soluble example and to a couple of approximately soluble examples.

II. SUPERSYMMETRIC QUANTUM MECHANICS

Witten[4] characterized SUSY quantum mechanics by the self-adjoint *supercharge* operators $Q_i (i = 1, 2, \ldots, N)$ satisfying the following graded algebra,

$$[Q_i, \mathcal{H}] = 0, \quad \{Q_i, Q_j\} = \mathcal{H}\delta_{ij}, \qquad i = 1, 2, \ldots, N \tag{2.1}$$

where $\mathcal{H}$ is the supersymmetric Hamiltonian. The algebra (2.1) implies $\mathcal{H} = Q_i^2$ for all i. Hence the Hamiltonian may be expressed in the form,

$$\mathcal{H} = \frac{2}{N}\sum_{i=1}^{N} Q_i^2. \tag{2.2}$$

Since this is positive semidefinite, spec $(\mathcal{H}) \geq 0$ in general. Let $|\psi>$ be the eigenstate belonging to the energy eigenvalue E, that is, $\mathcal{H}|\psi>= E|\psi>$. Then, $E \geq 0$.

In what follows, we consider the simplest non-trivial case $N = 2$. In this case, it is convenient to introduce the operators $Q = (Q_1 + iQ_2)/\sqrt{2}$ and $Q^\dagger = (Q_1 - iQ_2)/\sqrt{2}$. These operators obey the algebra,

$$\mathcal{H} = \{Q, Q^\dagger\}, \quad Q^2 = 0, \quad Q^{\dagger 2} = 0. \tag{2.3}$$

Obviously,

$$[Q, \mathcal{H}] = 0 \qquad [Q^\dagger, \mathcal{H}] = 0. \tag{2.4}$$

Let E_0 and $|\psi_0>$ be the ground state energy and the corresponding eigenstate, respectively. Then $\mathcal{H}|\psi_0>= E_0|\psi_0>$. SUSY is said to be *good* if

$$Q|\psi_0>= Q^\dagger|\psi_0>= 0. \tag{2.5}$$

This naturally implies $E_0 = 0$. Thus, if SUSY is a good symmetry, the ground state energy must necessarily be zero.

The supercharge operators may be realized in terms of bosonic and fermionic operators as

$$Q = A\xi \qquad Q^\dagger = A^\dagger\xi^\dagger \tag{2.6}$$

with

$$A = i\frac{1}{\sqrt{2m}}\hat{p} + \phi(\hat{q}) \qquad A^\dagger = -i\frac{1}{\sqrt{2m}}\hat{p} + \phi(\hat{q}). \tag{2.7}$$

Here $\hat{q}$ and $\hat{p}$ are the usual position and momentum operators obeying the commutation relation $[\hat{q}, \hat{p}] = i\hbar$; $\phi(\hat{q})$ is an operator-valued function of $\hat{q}$; and ξ and $\xi^\dagger$ are the Grassmann numbers having the properties,

$$\xi^2 = 0, \quad \xi^{\dagger 2} = 0, \quad \text{and} \quad \{\xi, \xi^\dagger\} = 1. \tag{2.8}$$

The supersymmetric Hamiltonian in (2.3) can now be put into the form,

$$\mathcal{H} = \frac{1}{2}\{A, A^\dagger\} + \frac{1}{2}[A, A^\dagger][\xi, \xi^\dagger], \tag{2.9}$$

or

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + \phi^2(\hat{q}) + \frac{i}{\sqrt{2m}}[\hat{p}, \phi(\hat{q})][\xi, \xi^\dagger]. \tag{2.10}$$

The Grassmann numbers satisfying (2.8) may further be represented by

$$\xi = \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \xi^\dagger = \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.11}$$

In this matrix representation, we have $[\xi, \xi^\dagger] = \sigma_3$, and hence the supersymmetric Hamiltonian is diagonalized:

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + \phi^2(\hat{q}) + \frac{i}{\sqrt{2m}}[\hat{p}, \phi(\hat{q})]\sigma_3, \tag{2.12}$$

or

$$\mathcal{H} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \tag{2.13}$$

with

$$H_+ = AA^\dagger = \frac{\hat{p}^2}{2m} + \phi^2(\hat{q}) + \frac{i}{\sqrt{2m}}[\hat{p}, \phi(\hat{q})], \tag{2.14}$$

$$H_- = A^\dagger A = \frac{\hat{p}^2}{2m} + \phi^2(\hat{q}) - \frac{i}{\sqrt{2m}}[\hat{p}, \phi(\hat{q})]. \tag{2.15}$$

In the q-representation, we have as usual $\phi(\hat{q}) \to \phi(q)$ and

$$\hat{p} \to -i\hbar\frac{d}{dq} \qquad [\hat{p}, \phi(\hat{q})] \to -i\hbar\phi'(q).$$

The two Hamiltonian elements, H_+ and H_-, are often referred to as the superpartners of each other. Following the recent practice, we shall call $\phi(\hat{q})$ the superpotential of SUSY quantum mechanics. The last terms of (2.14) and (2.15), which are of $\mathcal{O}(\hbar)$, stem from the presence of the fermionic variables in the charge operators (2.6). It may be viewed as a non-perturbative quantum correction originated from all fermion loops.

The eigenstates of the Hamiltonian (2-13) are doublets:

$$|\psi>= \begin{bmatrix} |\psi^{(+)}> \\ |\psi^{(-)}> \end{bmatrix}. \tag{2.16}$$

Suppose that the spectrum of $\mathcal{H}$ is discrete. Then, labelling the eigenvalues and the corresponding eigenstates by the index n, we have the eigenvalue equations,

$$H_+|\psi_n^{(+)}>= E_n^{(+)}|\psi_n^{(+)}>, \qquad H_-|\psi_n^{(-)}>= E_n^{(-)}|\psi_n^{(-)}>. \tag{2.17}$$

From (2.7), (2.14) and (2.15), we get

$$A^{\dagger}H_{+}|\psi_n^{(+)}>= H_{-}A^{\dagger}|\psi_n^{(+)}>= E_n^{(+)}A^{\dagger}|\psi_n^{(+)}>,$$
$$A^{\dagger}H_{-}|\psi_n^{(-)}>= H_{+}A^{\dagger}|\psi_n^{(-)}>= E_n^{(-)}A^{\dagger}|\psi_n^{(-)}>, \tag{2.18}$$

which imply for $E_n^{(+)} = E_n^{(-)} = E_n \neq 0$,

$$|\psi_n^{(+)}>= \frac{1}{\sqrt{E_n}}A|\psi_n^{(-)}>, \qquad |\psi_n^{(-)}>= \frac{1}{\sqrt{E_n}}A^{\dagger}|\psi_n^{(+)}>, \tag{2.19}$$

and

$$0 < \mathrm{spec}(H_{-}) = \mathrm{spec}(H_{+}), \tag{2.20}$$

except for the ground state energy E_0. As usual, we assume that the ground state is an eigenstate of H_{-}, that is, that $E_0^{(-)} = E_0$. Since SUSY means $E_0 = 0$, we have $A^{\dagger}|\psi_0>= 0$. In the q-representation, it reads

$$\left[\frac{\hbar}{\sqrt{2m}}\frac{d}{dq} + \phi(q)\right]\psi_0(q) = 0, \tag{2.21}$$

whose solution is

$$\psi_0(q) = \psi_0(q_0)\exp\left[-\frac{1}{\frac{\hbar}{\sqrt{2m}}}\int_{q_0}^{q}\phi(q)dq\right]. \tag{2.22}$$

If SUSY is a good symmetry, this solution must exist as the ground state of the system. In other words, for a good SUSY, this solution must be normalizable (or square-integrable).

The state function (2.22) is normalizable only when the integral of the real-valued superpotential $\phi(q)$ diverges as q tends to $\pm\infty$. Thus, normalizability of $\psi_0(q)$ implies that $\phi(q)$ has an odd number of zeros[4]. If the superpotential $\phi(q)$ has a definite parity, then its parity must be odd. Otherwise, SUSY is broken.

III. SEMI-CLASSICAL QUANTIZATION FORMULAS

SUSY quantum mechanics, rather independent of its original intent, may be viewed as a method to solve standard quantum mechanical problems. Suppose the Schrödinger equation for a one-dimensional potential problem in quantum mechanics is given by

$$H\psi(q) = \left\{-\frac{\hbar^2}{2m}\frac{d^2}{dq^2} + V(q)\right\}\psi(q) = E\psi(q), \tag{3.1}$$

with the potential $V(q)$ expressible in term of a function $\phi(q)$ as

$$V(q) = \phi^2(q) - \frac{\hbar}{\sqrt{2m}}\phi'(q) + E_0 \tag{3.2}$$

where E_0 is a constant. If we wish to estimate the energy eigenvalues of a one-dimensional system of this type, we may use the well-known quasi-classical quantization formula of Wentzel, Kramers, and Brillouin,

$$\int_a^b \sqrt{2m[E - V(q)]}dq = (n + \frac{1}{2})\pi\hbar, \tag{3.3}$$

where $q = a$ and $q = b$ are the turning points of the classical orbit, satisfying $V(a) = V(b) = E$, and $n \in N_0$. If we define a shifted energy spectrum $E_{-} = E - E_0$ and a shifted potential $V_{-}(q) = V(q) - E_0$, then we have $E - V(q) = E_{-} - V_{-}(q)$. Therefore, the WKB formula (3.3) may be expressed in the form,

$$\int_a^b \sqrt{2m[E_{-} - \phi^2(q) + \hbar\phi'(q)/\sqrt{2m}]}dq = (n + \frac{1}{2})\pi\hbar. \tag{3.4}$$

by which we can make a semi-classical estimate of the energy values E_{-} of the system (3.1).

Recently, Comtet, Bandrauk and Campbell[14] has proposed another quantization rule,

$$\sqrt{2m}\int_{q_L}^{q_R}\sqrt{2m[E_{-} - \phi^2(q)]}dq = n\pi\hbar. \tag{3.5}$$

where $\phi^2(q_L) = \phi^2(q_R) = E_{-}$. This is what we have been calling the CBC formula. Surprisingly, the CBC formula (3.5) provides exact energy spectra for the systems of so-called shape invariant potentials[8], including harmonic oscillators in one and three dimensions, the Morse oscillator, the Rosen-Morse oscillator, and the hydrogen atom among others. It is in contrast to the situation that the WKB formula (3-3) does not give exact results even for many of the shape invariant potentials unless the Langer-like ad hoc modification is made.

Let us consider, for instance, the radial harmonic oscillator with the usual effective potential,

$$V(r) = \frac{1}{2}m\omega^2r^2 + \frac{l(l+1)\hbar^2}{2mr^2}. \tag{3.6}$$

If we use the following superpotential,

$$\phi(r) = \sqrt{\frac{m}{2}}\omega r - \frac{(l+1)\hbar}{\sqrt{2m}r}, \tag{3.7}$$

then we obtain the shifted potential in the form,

$$V_{-}(r) = \frac{1}{2}m\omega^2r^2 + \frac{l(l+1)\hbar^2}{2mr^2} - (l + \frac{3}{2})\hbar\omega. \tag{3.8}$$

Substitution of the superpotential (3.7) into the CBC formula (3.5) leads to the energy spectrum $E_{-} = 2n\hbar\omega$ for the shifted oscillator. Apparently, the ground state energy (for $n = 0$) of the shifted oscillator is zero. If we employ the WKB formula (3.4), we can get the same result only after the well-known Langer replacement

$$l(l+1) \to (l+1/2)^2, \tag{3.9}$$

is applied. To find the energy spectrum for the original harmonic oscillator with (3.6), we calculate the ground state energy correction $E_0 = V(r - V_{-}(r) = (l + 3/2)\hbar\omega$. Then we add it to E_{-} to arrive at the well-known exact spectrum, $E_n = (2n + l + 3/2)\hbar\omega$.

Similarly, for any system whose spectrum is exactly obtainable from the WKB formula (3.4) with an appropriate Langer-like modification, the CBC formula (3.5) provides the same exact result with no ad hoc modification.

Since $l(l+1) = l'(l'+1)$ for $l' = -l-1$, let us try to replace l of (3.7) by $-l-1$ and construct an alternative superpotential,

$$\tilde{\phi}(r) = \sqrt{\frac{m}{2}}\omega r + \frac{l\hbar}{\sqrt{2m}r}. \tag{3.10}$$

The corresponding potential is not the same as (3.8). It is a harmonic oscillator potential shifted upward by $\tilde{E}_0 = (l - 1/2)\hbar\omega$,

$$\tilde{V}(r) = \frac{1}{2}m\omega^2r^2 + \frac{l(l+1)\hbar^2}{2mr^2} + (l - \frac{1}{2})\hbar\omega. \tag{3.11}$$

For this, the WKB formula (3.4) modified with the Langer replacement gives us a shifted energy spectrum $\tilde{E} = (2n+2l+1)\hbar\omega$. This time, the ground state energy correction is $\tilde{E}_0 = V(r) - \tilde{V}(r) = -(l - 1/2)\hbar\omega$. Thus, for the regular harmonic oscillator, we have again the same exact energy spectrum $E = \tilde{E} + \tilde{E}_0 = (2n + l + 1/2)\hbar\omega$. By contrast, the CBC formula (3.5) calculated for the superpotential (3.10) yields the spectrum $\tilde{E}' = (2n + 2l)\hbar\omega$ which differs from the result of the WKB formula by $\hbar\omega$. The CBC formula does not work for this case. In fact, neither $\tilde{E}$ nor $\tilde{E}'$ is zero for $n = 0$. This means that with the alternative superpotential (3.10) the ground state wave function is not normalizable and hence SUSY is broken. Consequently, the CBC formula (3.5) is no

longer applicable. The alternative superpotential is certainly incompatible with the CBC formula. However, this does not necessarily preclude the alternative choice. What is remarkable is that when SUSY is broken the WKB result with the Langer modification can be reproduced, without any *ad hoc* modification, from another quantization formula,

$$\sqrt{2m}\int_{q_L}^{q_R}[E_- - \phi^2(q)]^{1/2}dq = (n+\frac{1}{2})\pi\hbar \tag{3.12}$$

which is a hybrid of the WKB and the CBC formula. This hybrid formula can also produce the exact energy spectrum of the SUSY breaking Pöschl-Teller oscillator. In Section 5, we shall apply this formula to a couple of other examples.

IV. QUASI-CLASSICAL PATH INTEGRAL APPROACH

Although the CBC formula (3.5) works rather impressively, its origin is not very clear. The formula has been justified in relation with the WKB approximation. The main point of the justification is as follows. If the left hand side of the WKB formula (3.4) is expanded as

$$\begin{aligned}&\int_a^b \sqrt{2m(E_- - \phi^2(q) + \hbar\phi'(q)/\sqrt{2m})}dq \\ &= \sqrt{2m}\int_{q_L}^{q_R}[E_- - \phi^2(q)]^{1/2}dq + \tfrac{\hbar}{2}\int_{q_L}^{q_R}\frac{\phi'(q)\,dq}{\sqrt{E_- - \phi^2(q)}} + \mathcal{O}(\hbar^{3/2}),\end{aligned}$$

the second integral on the right hand side will be equal to $\pi\hbar/2$. As a result, the WKB formula can be reduced to the CBC formula within the semiclassical approximation. This only suggests that whenever the WKB formula is satisfactorily acceptable the CBC formula will satisfactorily work. However, the above justification can hardly be qualified as a derivation of the CBC formula. The second integral is not necessarily fixed to the aforementioned value. Furthermore, this does not explain why the CBC formula provides exact spectra for shape invariant potentials while the WKB formula gives only approximate results except for a few special examples unless appropriate ad hoc modifications are made. It is certainly desirable to derive the CBC formula from the first principle.

In an effort to derive the CBC formula, we shall analyze a path integral pertinent to SUSY quantum mechanics by a stationary phase method without relying upon the WKB formula. Actually, we shall arrive at the CBC formula (3.5) when SUSY is good and the hybrid formula (3.12) when SUSY is broken.

4.1 SUSY Path Integrals

First of all, we look for a SUSY path integral which can serve as our first principle. The time evolution generated by the SUSY Hamiltonian $\mathcal{H}$ is represented by the time evolution operator,

$$\mathcal{K}(t) = e^{-(i/\hbar)\mathcal{H}t}. \tag{4.1}$$

As has been seen in Section 2, it is possible to diagonalize the Hamiltonian $\mathcal{H}$. Hence the evolution operator $\mathcal{K}$ is diagnalizable as well. In particular, on the two component fermionic basis, we have

$$\mathcal{K}(t) = \begin{pmatrix} K_+ & 0 \\ 0 & K_- \end{pmatrix}, \tag{4.2}$$

where

$$K_\pm(t) = e^{-(i/\hbar)H_\pm t}. \tag{4.3}$$

In the q-representation, the components of the evolution operator yield the propagators for the two separate Hamiltonians $H_\pm$:

$$K_\pm(q'',q';t'',t') = < q''|e^{-(i/\hbar)H_\pm(t''-t')}|q' > . \tag{4.4}$$

Since we have assumed that the ground state is an eigenstate of the Hamiltonian H_- of (2.15), we are interested in a system whose Hamiltonian is given by H_-. Hence, now on, we shall consider

only the propagator $K(q'',q';t'',t') = K_-(q'',q';t'',t')$. The matrix elements on the right hand side of (4.4) for H_- can be written in a path integral form:

$$K(q'',q';t'',t') = \int \exp\left[\frac{i}{\hbar}\int_{t'}^{t''} L_-(\dot{q},q)dt\right]\mathcal{D}q. \tag{4.5}$$

Here,

$$L_- = \frac{1}{2}m\dot{q}^2 - \phi^2(q) + \frac{\hbar}{\sqrt{2m}}\phi'(q) \tag{4.6}$$

is the Lagrangian corresponding to the Hamiltonian H_-. Now, we choose (4.5) as our starting path integral for the propagator in SUSY quantum mechanics.

Alternatively, we may construct formally Feynman's path integral from the Lagrangian,

$$\mathcal{L} = \frac{1}{2}m\dot{q}^2 - \phi^2(q) + i\xi\dot{\xi}^\dagger + \frac{\hbar}{\sqrt{2m}}\phi'(q)[\xi,\xi^\dagger], \tag{4.7}$$

which corresponds to $\mathcal{H}$. The resultant path integral naturally includes not only coordinate variables but also the Grassmann variables. After carrying out path integration over the Grassmann variables, diagonalizing the result on the two component basis, and taking the second diagonal element, we can reach the same path integral as the one we have chosen.

4.2 Quasi-Classical Calculation

Let us write the path integral (4.5) in the form,

$$K(x'',x';T) = \int_{x'=q(0)}^{x''=q(T)} e^{(i/\hbar)S[q(t)]}\mathcal{D}q(t) \tag{4.8}$$

where

$$S(x'',x';T) = \int_0^T \left[\frac{m}{2}\dot{q}^2 - \phi^2(q) + \frac{\hbar}{\sqrt{2m}}\phi'(q)\right]dt. \tag{4.9}$$

As is well-known, the WKB quantization rule (3.4) can be obtained for the system with H_- by counting the contributions from the classical paths along which $\delta S(x'',x';T) = 0$. In contrast, to derive the CBC formula, we employ the quasi-classical approach. By "the quasi-classical approach," we mean the stationary phase approximation that counts contributions only from the paths $q(t) = x(t)$ along which the "tree" action functional,

$$S_{\rm tree}(x'',x';T) = \int_0^T \left[\frac{m}{2}\dot{q}^2 - \phi^2(q)\right]dt, \tag{4.10}$$

is stationary. Here the contributions of $\mathcal{O}(\hbar)$ from the fermion loops are neglected. This approximation is based on the observation that contributions from the classical paths at the tree level will dominate in the path integral (4.8). Under the condition $\delta S_{\rm tree}[q(t)] = 0$, we are considering a Lagrangian of the form,

$$\tilde{L}(\dot{x},x) = \frac{m}{2}\dot{x}^2 - \tilde{V}(x) \tag{4.11}$$

where $\tilde{V}(x) = \phi^2(x)$ is the effective potential at the tree level. The "quasi-classical" paths $x(t)$ are solutions of the Euler-Lagrange equations based on this Lagrangian. The constant of motion along the quasi-classical path is the quasi-classically approximate energy $\tilde{E} = m\dot{x}^2/2 + \tilde{V}(x)$ from which follows

$$p_{\rm qc}(x) = m\dot{x} = \pm\sqrt{2m(\tilde{E} - \tilde{V}(x))} \tag{4.12}$$

where the + and − signs signify the motions to the right and left, respectively.

In carrying out the quasi-classical calculation, we first expand the action (4.9) about the quasi-classical path $x(t)$ by setting $q(t) = x(t) + \eta(t)$:

$$S(q'',q';T) \simeq S_{\rm qc}(x'',x';T) + \delta^2 S(T) + S_F(x'',x';T) + \cdots . eqno(4.13)$$

Here, the first term is the action evaluated along the quasi-classical path $x(t)$:

$$S_{\rm qc}(x'',x';T)=\int_0^T \tilde{L}(\dot{x}(t),x(t))dt \tag{4.14}$$

the second term is the second variation of (4.9):

$$\delta^2 S(T)=\int_0^T (\frac{m}{2}\dot{\eta}^2-\frac{1}{2}\tilde{p}^2(x)\eta^2)dt \tag{4.15}$$

and the third term stems from the fermionic correction:

$$S_F(x'',x';T)=\frac{\hbar}{\sqrt{2m}}\int_0^T \phi'(x(t))dt. \tag{4.16}$$

For convenience, we introduce the quantity

$$\varphi=\frac{1}{\hbar}S_F(x'',x';T) \tag{4.17}$$

and call it the *fermionic phase*. It can be expressed as

$$\varphi=\frac{1}{\sqrt{2m}}\int_0^T \phi'(x(t))dt=\frac{1}{2}\int_{x'}^{x''}\frac{\phi'(x)}{\pm\sqrt{\tilde{E}-\phi^2(x)}}dx \tag{4.18}$$

where the last integral is to be evaluated along the quasi-classical path $x(t)$.

In this approximation, the path integral (4.8) becomes Gaussian and can explicitly be calculated[20], the result being

$$\begin{aligned}K(x'',x';T)&\simeq\sqrt{\frac{i}{2\pi\hbar}}\sum_{\text{fixed }T}\sqrt{\frac{\partial^2 S_{\rm qc}}{\partial x''\partial x'}}\exp\{(i/\hbar)S_{\rm qc}+i\varphi\}\\&=\sqrt{\frac{i}{2\pi\hbar}}\sum_{\text{fixed }T}\sqrt{\left|\frac{\partial^2 S_{\rm qc}}{\partial x''\partial x'}\right|}\exp\{(i/\hbar)S_{\rm qc}+i\varphi-i(\pi/2)\mu\}.\end{aligned} \tag{4.19}$$

The second line indicates that the square root of the van Vleck-Pauli-Morette determinant in the sum must be carefully evaluated. The Morse index μ is the number of times this determinant changes its sign along the path. It is identical with the number of focal points (counted with their multiplicity) on the path[20].

The sum in (4.19) is evaluated over all quasi-classical paths leading from x' to x'' for a *fixed* time duration T. Unfortunately, we cannot solve the classical problem for a fixed T and hence we are unable to perform the summation of (4.19). To circumvent this difficulty we go over from the propagator to the energy dependent Green function

$$G(x'',x';E)=\frac{1}{i\hbar}\int_0^\infty K(x'',x';T)e^{(i/\hbar)ET}dT. \tag{4.20}$$

This integral can also be evaluated by the stationary phase method. To this end, we shall closely follow the procedure given in Schulman's book[20] which is based on the original work of Gutzwiller[19]. The steepest decent method selects only the values of T for which $E=-\partial S_{\rm qc}/\partial T$. The Green function is then approximated by[20]

$$G(x'',x';E)\simeq\frac{1}{i\hbar}\sqrt{|D|}\sum_{\text{fixed }E}\exp\{(i/\hbar)W+i\varphi-i(\pi/2)\nu\} \tag{4.21}$$

where the sum is now taken over all classical paths $x(t)$ from x' to x'' with a *fixed energy* E. In the above $W(x'',x';E)$ is Hamilton's characteristic function,

$$W(x'',x';E):=S_{\rm qc}(x'',x';T)+ET=\int_{x'}^{x''}p_{\rm qc}(x)dx. \tag{4.22}$$

The last integral is again evaluated along the quasi-classical path $x(t)$. The determinant D appearing in (4.21) can be explicitly calculated and given by[20]

$$|D|=\frac{m}{2}[(E-V_{\rm qc}(x''))(E-V_{\rm qc}(x'))]^{-1/2}=\frac{m^2}{|p_{\rm qc}(x'')p_{\rm qc}(x')|}. \tag{4.23}$$

This gives rise to the so-called Maslov index ν which is for a one-dimensional problem identical with the number of turning points along the path.[20] In fact, as long as $\partial E/\partial T>0$ the Maslov index ν is identical with the Morse index μ.[20]

For each quasi-classical trajectory with the same fixed energy value E we have to find W, φ and ν in order to carry out the summation in (4.15). For this purpose, we divide the set of all paths into four classes.[20] The paths belonging to the first class are those which leave x' to the right and reach x'' from the left. This class may be characterized by the ordered pair of indices (R,L). Similarly, the remaining three classes are described by the ordered pairs of indices, (L,L), (R,R) and (R,L). Within each class a particular path is uniquely described by the number of full cycles. The class will be indicated by a superscript in brackets and the number of full cycles by a subscript, e.g. $\varphi_k^{(i)}$. Let us denote the classical turning points by x_L and x_R, respectively, and define

$$w(x)=\int_{x_L}^{x}\sqrt{2m(E-V_{\rm qc}(q))}dq,\qquad a(x)=\arcsin\frac{\phi(x)}{\sqrt{E}}. \tag{4.24}$$

With these we can calculate all relevant quantities appearing in the sum (4.21). For the shortest path in each class we have:

Class (R,L): This is the direct path from x' to x'' with no reflection, for which

$$\begin{aligned}W_0^{(1)}&=\int_{x'}^{x''}\sqrt{2m(E-\phi^2(x))}dx=w(x'')-w(x')\\\nu_0^{(1)}&=0\\\varphi_0^{(1)}&=\tfrac{1}{2}\int_{x'}^{x''}\frac{\phi'(x)}{\sqrt{E-\phi^2(x)}}dx=\tfrac{1}{2}[a(x'')-a(x')]\end{aligned} \tag{4.25}$$

Class (L,L): This path starts from x', is reflected once at x_L, and reaches x'', which has

$$\begin{aligned}W_0^{(2)}&=w(x'')+w(x')\\\nu_0^{(2)}&=1\\\varphi_0^{(2)}&=\tfrac{1}{2}[a(x'')+a(x')]-a(x_L)\end{aligned} \tag{4.26}$$

Class (R,R): This path starts at x', passes x'' once, is reflected at x_R, and reaches x'' again from right, having

$$\begin{aligned}W_0^{(3)}&=2w(x_R)-w(x'')-w(x')\\\nu_0^{(3)}&=1\\\varphi_0^{(3)}&=a(x_R)-\tfrac{1}{2}[a(x'')+a(x')]\end{aligned} \tag{4.27}$$

Class (L,R): This path, during its passage from x' to x'', is reflected first at x_L and later at x_R, for which

$$\begin{aligned}W_0^{(4)}&=2w(x_R)-w(x'')+w(x')\\\nu_0^{(4)}&=2\end{aligned} \tag{4.28}$$

$$\varphi_0^{(4)}=a(x_R)-a(x_L)-\tfrac{1}{2}[a(x'')-a(x')]. \tag{4.29}$$

From these results we can obtain the corresponding expressions for an arbitrary path (making k cycles, $k\in N_0$) in each class:

$$\begin{aligned}W_k^{(i)}&=W_0^{(i)}+2kw(x_R)\\\nu_k^{(i)}&=\nu_0^{(i)}+2k\\\varphi_k^{(i)}&=\varphi_0^{(i)}+k[a(x_R)-a(x_L)].\end{aligned} \tag{4.30}$$

Now we are ready to perform the summation in (4.21) at least partially. First, we rewrite the sum as

$$\sum_{\text{fixed } E} \exp\{(i/\hbar)W + i\varphi - i(\pi/2)\nu\} = \sum_{i=1}^{4} \exp\{(i/\hbar)W_0^{(i)} + i\varphi_0^{(i)} - i(\pi/2)\nu_0^{(i)}\} \times \sum_{k=0}^{\infty} \exp\{ik[2w(x_R)/\hbar + a(x_R) - a(x_L) - \pi]\}. \tag{4.31}$$

The second sum is a geometric series which is easily computed. Therefore, in the quasi-classical approximation, the Green function can be expressed in the form,

$$G(x'', x'; E) \simeq \frac{m}{i\hbar\sqrt{|p_{qc}(x'')p_{qc}(x')|}} \frac{\sum_{i=1}^{4} \exp\{(i/\hbar)W_0^{(i)} + i\varphi_0^{(i)} - i(\pi/2)\nu_0^{(i)}\}}{1 - \exp\{i[2w(x_R)/\hbar + a(x_R) - a(x_L) - \pi]\}}. \tag{4.32}$$

From this Green function, we can find the energy spectrum and the energy eigenfunctions.

4.3 Quasi-Classical Quantization Rules

The poles of the Green function (4.32) give rise to the quasi-classical energy spectrum, which occur when

$$w(x_R) \equiv \int_{x_L}^{x_R} p_{qc}(x)dx = \hbar\pi(n + \frac{1}{2} - \frac{a(x_R) - a(x_L)}{2\pi}), \quad n \in N_0 \tag{4.33}$$

For the explicit values of $a(x_L)$ and $a(x_R)$,we have to go back to the definition of the turning points, $\phi^2(x_L) = \phi^2(x_R) = E$. Since we may assume for our discussion without loss of generality that $\phi(x_R) > 0$, we have the following two situations at the end points:

Case I: $\phi(x_R) = \phi(x_L) = \sqrt{E}$

In this case, SUSY is a good symmetry and we have $a(x_R) = -a(x_L) = \pi/2$. This leads us to the expression

$$\int_{x_L}^{x_R} p_{qc}(x)dx = n\pi\hbar, \quad n \in N_0 \tag{4.34}$$

which is nothing but the formula of Comtet, Bandrauk and Campbell[14]. This CBC formula gives the exact ground state energy $E_0 = 0$ for $n = 0$. Here we wish to emphasize that if SUSY is a good symmetry the quasi-classical calculation of Feynman's path integral naturally leads us to the CBC formula. The contribution of the fermionic phase φ is also very clear in this approach.

Case II: $\phi(x_R) = \phi(x_L) = \sqrt{E}$.

In this case, (4.32) gives us $a(x_R) = a(x_L) = \pi/2$. Thus, (4.33) leads us

$$\int_{x_L}^{x_R} p_{qc}(x)dx = (n + \frac{1}{2})\pi\hbar, \quad n \in N_0, \tag{4.35}$$

which is indeed identical with the WKB-like hybrid formula (3.12). Of course, it coincides with the usual WKB result for the "tree" Hamiltonian $H_{tree} = \hat{p}^2/2m + \phi^2(\hat{q})$. Since $\phi(x)$ has an even number of zeros for $x_L < x < x_R$, SUSY is broken as has been discussed in Section 2. Therefore, the quasi-classical approximation for the broken SUSY case is the same as the standard WKB expression for H_{tree}. However, we are dealing with the system of H_- rather than H_{tree}. Thus, the broken SUSY formula (4.35) apparently differs, when applied to quantum mechanics, from the standard WKB formula. Earlier Eckhardt[21] has arrived at the same formula as (4.35) from the Maslov index consideration without noticing the relation between the fermionic phase and broken SUSY.

4.4 Quasi-Classical Energy Eigenfunctions

The energy eigenfunctions can be obtained from the residues of (4.32) at the poles $E = E_r$. To calculate the residues, we have to perform the remaining summation in the Green function (4.32). Again we treat the two cases of SUSY and broken SUSY separately.

Case I: (good SUSY): In this case, the remaining sum in (4.32) can be reduced to a product of two cosine functions,

$$\sum_{i=1}^{4} \exp\{\frac{i}{\hbar}W_0^{(i)} + i\varphi_0^{(i)} - \frac{i\pi}{2}\nu_0^{(i)}\} = 4\cos[\frac{w(x')}{\hbar} + \frac{a'(x')}{2}]\cos[\frac{w(x'')}{\hbar} + \frac{a(x'')}{2}].$$

and hence the residues of $G(x'', x'; E)$ at the pole $E = E_n$ are given by

$$\text{Res}G(x'', x'; E)|_{E=E_n} = \frac{4m}{\tau_n} \frac{\cos(w(x')/\hbar + a(x')/2)}{\sqrt{|p_{qc}(x')|}} \frac{\cos(w(x'')/\hbar + a(x'')/2)}{\sqrt{|p_{qc}(x')|}} \tag{4.36}$$

where

$$\tau_n = \int_{x_L}^{x_R} \sqrt{\frac{2m}{E_n - \phi^2(q)}}dq \tag{4.37}$$

is the period of the bounded classical motion with energy E_n. From this, we obtain the normalized wave function for a fixed energy eigenvalue $E = E_n > 0$ (that is, for $n > 0$):

$$\Psi_n(x) = \sqrt{\frac{4m}{\tau_n}}|p_{qc}(x)|^{-1/2}\cos(\frac{1}{\hbar}\int_{x_L}^{x} p_{qc}(q)dq + \frac{1}{2}\arcsin\frac{\phi(x)}{\sqrt{E_n}}), \quad n \in N. \tag{4.38}$$

Here, $(x_L < x < x_R)$.

Case II (broken SUSY): The sum in (4.32) becomes in this case a product of two sine functions, and the corresponding residues are given by

$$\text{Res}G(x'', x'; E)|_{E=E_n} \simeq \frac{4m}{\tau_n} \frac{\sin(w(x')/\hbar + a(x')/2)}{\sqrt{|p_{qc}(x')|}} \frac{\sin(w(x'')/\hbar + a(x'')/2)}{\sqrt{|p_{qc}(x'')|}}. \tag{4.39}$$

From this, again, we obtain the wave function ($E = E_n > 0$):

$$\Psi_n(x) = \sqrt{\frac{4m}{\tau_n}}|p_{qc}(x)|^{-1/2}\sin(\frac{1}{\hbar}\int_{x_L}^{x} p_{qc}(q)dq + \frac{1}{2}\arcsin\frac{\phi(x)}{\sqrt{E_n}}), \quad n \in N_0, \tag{4.40}$$

which is also properly normalized.

These results may be compared with the usual WKB wave function for the tree Hamiltonian:

$$\Phi_n(x) = \sqrt{\frac{4m}{\tau_n}}|p_{qc}(x)|^{-1/2}\sin(\frac{1}{\hbar}\int_{x_L}^{x} p_{qc}(q)dq + \frac{\pi}{4}). \tag{4.41}$$

What makes our quasi-classical energy eigenfunctions different from this WKB result is the presence of the extra non-constant phase $\frac{1}{2}\arcsin(\phi(x)/\sqrt{E_n})$ in the place of a constant phase $\pi/4$.

This phase in case I (good SUSY) starts at the value $-\pi/4$ for x_L and ends with the value $\pi/4$ at x_R. In case II (broken SUSY), the phase changes from $\pi/4$ to zero and back to $\pi/4$ as x varies from x_L to x_R. For broken SUSY the wave function is similar to the WKB wave function (4.41) given as a "sine" function. When SUSY is good, the wave function is a "cosine" function. This is contrary to some recent claims which suggested the sine as the correct expression[17] but is in agreement with that of ref.[18].

V. BROKEN SUSY FORMULA

The quantization rule of Comtet, Bandrauk and Campbell has been well-studied, but the significance of the WKB-like hybrid formula derived for the case of broken SUSY is not clear. Here we wish to examine the role of the broken SUSY formula (4.35) in quantum mechanics by solving a couple of examples.

In Section 3, we have already seen that the hybrid formula (4.35) gives the exact energy spectrum for the radial harmonic oscillator if a SUSY breaking superpotential is utilized. As we have also

mentioned earlier, we can make a similar argument for the Pöschl-Teller oscillator. As the CBC formula provides an exact energy spectrum of this system when a SUSY preserving superpotential is used, so does the broken SUSY formula when a SUSY breaking superpotential is chosen.

It is natural to ask if this new quantization condition can serve significantly as a possible approximation formula for a non-shape invariant system in much the same fashion that the WKB formula does. In order to seek an answer to this question, we examine the superpotentials $\phi(q) = q|q|$ and $\phi(q) = q^3$ both of which retain SUSY good. In Figures 1 a) and 1 b), we compare the approximate energy values from the WKB formula and the CBC formula with the exact energies evaluated numerically from the Schrödinger equation. While CBC gives exact results for the ground state, WKB seems to give better results for the first exited state. However, both formulas provide approximate energy values which fluctuate above and below the exact results. It is difficult to judge which approximation is better.

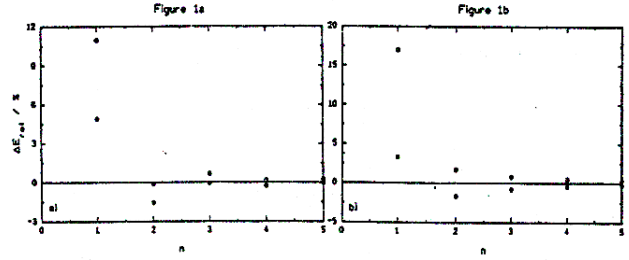

Figure 1: Relative errors for the WKB (•) and CBC (○) quantization. For a) $\phi(q) = q|q|$, and for b) $\phi(q) = q^3$. In both cases SUSY is good.

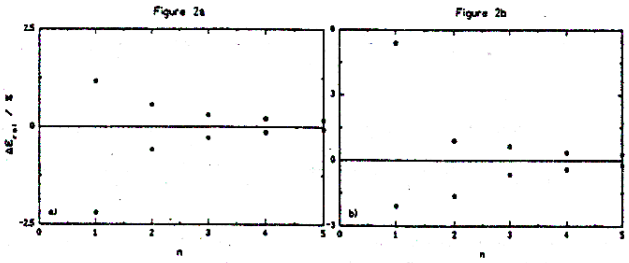

Figure 2: Relative errors for the WKB (•) and the WKB-like (○) quantization. For a) $\phi(q) = q^2$, and for b) $\phi(q) = |q|^3$. In both cases SUSY is broken.

Next, we study the broken-SUSY case by taking two superpotentials, $\phi(q) = q^2$ and $\phi(q) = |q|^3$. Since SUSY is broken, the CBC formula (3.5) is no longer applicable. Thus, we use the WKB formula (3.4) and the broken-SUSY formula (3.12) to evaluate the energy values for the two systems. In Figures 2 a) and 2 b), the WKB values and the broken SUSY values are compared with the numerical result from the Schrödinger equation. In each of the two examples, we observe that the WKB formula (3.5) always underestimates the energy values whereas the broken-SUSY formula (3.12) always overestimates them. If this observation can be shown to be generally true for any broken-SUSY problem, the WKB approximation (3.5) provides a lower bound and the broken SUSY quantization (3.12) gives an upper bound for the exact eigenvalues. Hence, taking average of the two approximations may result in an improved quasi-classical energy spectrum. However, even if this expectation turns out to be negative, the broken-SUSY approximation will remain as good as

the WKB approximation.

VI. CONCLUDING REMARKS

We have made a quasi-classical analysis of the path integral in SUSY quantum mechanics, and derived the quantization rule of Comtet, Bandrauk and Campbell. As a by-product, we have obtained another quantization rule which is applicable only when SUSY is broken. We have found that the new formula reproduces exact spectra for some of the shape-invariant potential systems, and that as an approximation formula it is as good as the WKB formula. There is an indication that while the WKB formula underestimates the broken SUSY formula overestimates when the problem is not exactly soluble.

Finally, we wish to point out the fact that when SUSY is good the fermionic phase φ appearing in (4.21) precisely cancels the Maslov index for all paths, i.e., $\varphi_k^{(\prime)} = (\pi/2)\nu_k^{(\prime)}$. This seems to indicate an interesting connection between the Maslov theory in ordinary quantum mechanics and SUSY quantum mechanics. Recently, it has been argued[22] that there is a close connection between the Maslov correction and the parity of the system in question. The number of zeros of a superpotential which is a key to decide whether SUSY is good or not is indeed directly related to the parity of the superpotential.

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